Homework 2
Geometry

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Proposition 0.1 (Exercise A.46). *Let X, Y be topological spaces.*

1. *If $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact.*
2. *If X is compact and $f : X \to \mathbb{R}$ is continuous, then f is bounded and attains its maximum and minimum values on X.*
3. *Every closed subset of a compact spaces is compact.*
4. *Every compact subset of a Hausdorff space is closed.*

Proof. First we prove (1). Let $f : X \to Y$ be continuous and X be compact. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of $f(X)$. Then

$$f(X) \subset \bigcup_\alpha U_\alpha \implies f^{-1}(f(X)) \subset f^{-1}\left(\bigcup_\alpha U_\alpha\right)$$

Since $X \subset f^{-1}(f(X))$ and $f^{-1}(f(X)) \subset X$, these sets are equal. Note also that the preimage of a union is the union of preimages, so

$$X \subset \bigcup_\alpha f^{-1}(U_\alpha)$$

Since f is continuous $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is an open cover of X. Since X is compact, there is a finite subcover of X, $\{f(U_i)\}_{i=1}^n$. Then

$$X \subset \bigcup_{i=1}^n f^{-1}(U_i) \implies f(X) \subset f\left(\bigcup_{i=1}^n f^{-1}(U_i)\right) = \bigcup_{i=1}^n f(f^{-1}(U_i)) \subset \bigcup_{i=1}^n U_i$$

since $f(f^{-1}(U_i)) \subset U_i$ for each i. Thus $\{U_i\}_{i=1}^n$ is an open cover for $f(X)$. Hence every open cover of $f(X)$ can be reduced to a finite subcover, so $f(X)$ is compact.

Now we prove (2). Let $f : X \to \mathbb{R}$ be continuous and X be compact. Then by (1), $f(X) \subset \mathbb{R}$ is compact. By the Heine-Borel theorem, $f(X)$ is closed and bounded, thus f is bounded. Since $f(X)$ is closed, it includes all limit points, in particular, it includes $\sup f(X)$ and $\inf f(X)$. Thus f attains its maximum and minimum values on X.

Now we prove (3). Let X be a compact space and let $C \subset X$ be closed. Let $\{U_\alpha\}_{\alpha \in A}$. Then $\{U_\alpha\} \cup (X \setminus C)$ is an open cover for X, so it has a finite subcover (by compactness of X). Such a subcover must include at most finitely many U_α; index the remaining U_α by i as $\{U_i\}_{i=1}^n$. We claim that $\{U_i\}$ is a cover for C, since the only other possible set in this finite subcover for X is $X \setminus C$, which has empty intersection with C. Hence $C \subset \bigcup_{i=1}^n U_i$. Hence $\{U_i\}$ is a finite subcover of C of the original cover $\{U_\alpha\}$, so any open cover of C has a finite subcover. Hence C is compact.

Now we prove (4). Let X be a Hausdorff topological space, and let $A \subset X$ be compact. We will show that A is closed by showing that $X \setminus A$ is open. Let $x \in X \setminus A$. Then for each $a \in A$, there exist open neighborhoods U_a, V_a such that $a \in U_a, x \in V_a, U_a \cap V_a = \emptyset$ (by Hausdorff property of X). Then $A \subset \bigcup_{a \in A} U_a$, so $\{U_a\}$ is an open cover for A, so we can find a finite subcover $\{U_{a_i}\}_{i=1}^n$ (by compactness of A). Let $V = \bigcap_{i=1}^n V_{a_i}$. Then V and A are disjoint, since

$$y \in V \implies \forall i, y \in V_{a_i} \implies \forall i, y \notin U_{a_i} \implies y \notin \bigcup_i U_{a_i} \implies y \notin A$$

Thus $V \cap A = \emptyset$. And V is open, since it is a finite intersection of open sets. Finally, V contains X since each V_a contains x. Hence V is an open neighborhood of x contained within $X \setminus A$. Since x was arbitrary, this means that $X \setminus A$ is open, hence A is closed. □

Lemma 0.2 (for Exercise 1-3). *Let $\phi : X \to Y$ be an open, continuous, and surjective map and $\mathcal{B}$ a basis for X. Then $\phi(\mathcal{B})$ is a basis for Y.*

Proof. Let $\mathcal{B} = \{B_\alpha\}\alpha \in A$. Then $\bigcup_\alpha B_\alpha = X$ and each $U \subset X$ open can be expressed as $\bigcup_{i \in I} B_i$. Then

$$Y = \phi(X) = \phi\left(\bigcup_\alpha B_\alpha\right) = \bigcup_\alpha \phi(B_\alpha)$$

Thus the collection $\{\phi(B_\alpha)\}_{\alpha \in A}$ covers Y. Let $V \subset Y$ be open. Then $\phi^{-1}(V) \subset X$ is open, so

$$\phi^{-1}(V) = \bigcup_{i \in I} B_i$$
$$\phi(\phi^{-1}(V)) = \phi\left(\bigcup_{i \in I} B_i\right)$$
$$V = \bigcup_{i \in I} \phi(B_i)$$

Thus V can be written as a union of B_i. □

Proposition 0.3 (Exercise 1-3). *A locally Euclidean Hausdorff space is a topological manifold if and only if it is σ-compact.*

Proof. First suppose that X is a topological manifold (then X is locally Euclidean and Hausdorff by definition). We need to express X as a union of countably many compact subspaces to show that it is σ-compact. By Lemma 1.10, X has a countable basis of precompact coordinate balls, $\{U_i\}_{i=1}^{\infty}$. For each i, the closure of U_i is compact and contains U_i, so the collection $\{\overline{U_i}\}_{i=1}^{\infty}$ is a countable cover of X by compact subspaces. Hence X is σ-compact.

Now suppose that X is a σ-compact, locally Euclidean Hausdorff space. We must show that X is second-countable, that is, we must find a countable basis for X. Since X is σ-compact, we can write X as a union of countably many compact subspaces, $X = \bigcup_{i=1}^{\infty} K_i$. For each $p \in X$, there is a local chart (U_p, ϕ_p) with $p \in U_p$ and where U_p is homeomorphic to the unit ball in $\mathbb{R}^n$ (because X is locally Euclidean). For each i, the union $\bigcup_{p\in X} U_p$ is an open cover of K_i, so we can find a finite subcover (because K_i is compact),

$$K_i \subset \bigcup_{j=1}^{n} U_{ij}$$

Since $\mathbb{R}^n$ is second-countable, there is a countable basis $\{B_{ijk}\}_{k=1}^{\infty}$ for each $\phi_{ij}(U_{ij}) \subset \mathbb{R}^n$. Let $V_{ijk} = \phi_{ij}^{-1}(B_{ijk})$. Notice then that $U_{ij} \subset \bigcup_k V_{ijk}$. We claim that

$$\{V_{ijk} : i, j, k \geq 1\}$$

is a countable basis for X. It is clearly countable. Each V_{ijk} is open since it is a preimage of an open set in $\mathbb{R}^n$. It is not hard to see that they cover X, since

$$X = \bigcup_i K_i \subset \bigcup_i \bigcup_j U_{ij} \subset \bigcup_i \bigcup_j \bigcup_k V_{ijk}$$

Finally, we need to show that any open set $\mathcal{O} \subset X$ can be written as a union of V_{ijk}. Let $\mathcal{O} \subset X$ be open. For all i, j, k, the set $\mathcal{O} \cap V_{ijk}$ is open because V_{ijk} is open. Then the union

$$\bigcup_{i,j,k} (\mathcal{O} \cap V_{ijk})$$

is a union of open sets, which makes it open. It is obvious that this union is contained in $\mathcal{O}$. It also contains $\mathcal{O}$, since the V_{ijk} cover X. Thus we have

$$\mathcal{O} = \bigcup_{i,j,k} (\mathcal{O} \cap V_{ijk})$$

□

Proposition 0.4 (Exercise 1-7a). *Let $N = (0, 0, \ldots, 1) \in S^n \subset \mathbb{R}^{n+1}$ denote the north pole and $S = (0, 0, \ldots, -1)$ be the south pole. We define the stereographic projection $\sigma : S^n \setminus N \to \mathbb{R}^n$ by*

$$\sigma(x^1, \ldots x^{n+1}) = \frac{(x^1, \ldots x^n)}{1 - x^{n+1}}$$

and we define $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in S^n \setminus S$. Then for any $x \in S^n \setminus N$, $(\sigma(x), 0)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$. Similarly, $\tilde{\sigma}(x)$ is the point where the line through S and x intersects the same subspace.

Proof. To show this, we show that we can write $(\sigma(x), 0)$ as a linear combination of $x - N$ and x. Let $a = x^{n+1}/(1 - x^{n+1})$. Then as a preliminary, we calculate

$$\begin{aligned} a + 1 &= \frac{1}{1 - x^{n+1}} \\ a(x^{n+1} - 1) + x^{n+1} &= \frac{x^{n+1}(x^{n+1} - 1)}{1 - x^{n+1}} + x^{n+1} = -x^{n+1} + x^{n+1} = 0 \end{aligned}$$

Now we can show that $(\sigma(x), 0) = a(x - N) + x$.

$$\begin{aligned} a(x - N) + x &= a(x^1, \dots x^{n+1} - 1) + (x^1, \dots x^{n+1}) \\ &= \big((a + 1)x^1, \dots (a + 1)x^n, a(x^{n+1} - 1) + x^{n+1}\big) \\ &= \left(\frac{1}{1 - x^{n+1}}(x^1, \dots x^n), 0\right) \\ &= (\sigma(x), 0) \end{aligned}$$

Thus x, N, and $(\sigma(x), 0)$ are collinear, and clearly $(\sigma(x), 0)$ is in the linear subspace where $x^{n+1} = 0$.

Now we show that $x, S, \tilde{\sigma}(x)$ are collinear. Now let $a = -x^{n+1}/(1 + x^{n+1})$. Then

$$\begin{aligned} a + 1 &= 1/(1 + x^{n+1}) \\ a(x^{n+1} + 1) + x^{n+1} &= \frac{-x^{n+1}(1 + x^{n+1}}{1 + x^{n+1}} + x^{n+1} = -x^{n+1} + x^{n+1} = 0 \end{aligned}$$

so we can compute

$$\begin{aligned} a(x - S) + x &= a(x^1, \dots x^{n+1} + 1) + (x^1, \dots x^{n+1}) \\ &= \big((a + 1)x^1, \dots (a + 1)x^n, a(x^{n+1} + 1) + x^{n+1}\big) \\ &= \left(\frac{1}{1 + x^{n+1}}(x^1, \dots x^n), a(x^{n+1} + 1) + x^{n+1}\right) \\ &= \frac{(x^1, \dots x^n, 0)}{1 + x^{n+1}} \\ &= (-\sigma(-x), 0) \\ &= (\tilde{\sigma}(x), 0) \end{aligned}$$

Thus $(\tilde{\sigma}(x), 0)$ is collinear with x, S. □

Proposition 0.5 (Exercise 1-7b)**.** *The stereographic projection σ is a bijection, with inverse σ^{-1} given by*

$$\sigma^{-1}(x) = \sigma^{-1}(x^1, \dots x^n) = \frac{(2x^1, \dots, 2x^n, |x|^2 - 1)}{|x|^2 + 1} =$$

Proof. Let σ^{-1} be as stated above. We will show that $\sigma \circ \sigma^{-1}$ and $\sigma^{-1} \circ \sigma$ are the identity on their respective domains. First, let $x \in S^n \setminus \{0\}$. Let $x = (x^1, \dots x^{n+1} \in S^n \setminus \{N\}$. As a preliminary calculation, we compute $|\sigma(x)|^2$, since this term arises in computing $\sigma^{-1} \circ \sigma(x)$. (Note that $|x| = 1$ since x is on S^n.)

$$
\begin{aligned}
|\sigma(x)|^2 &= \frac{(x^1)^2 + \ldots + (x^n)^2)}{(1 - x^{n+1})^2} \\
&= \frac{(x^1)^2 + \ldots + (x^n)^2 + (x^{n+1})^2 - (x^{n+1})^2}{(1 - x^{n+1})^2} \\
&= \frac{|x| - (x^{n+1})^2}{(1 - x^{n+1})^2} \\
&= \frac{1 - (x^{n+1})^2}{(1 - x^{n+1})^2}
\end{aligned}
$$

Now we can compute $\sigma^{-1} \circ \sigma(x)$ directly.

$$
\begin{aligned}
\sigma^{-1} \circ \sigma(x) &= \sigma^{-1}\left(\frac{(x^1, \ldots x^n)}{1 - x^{n+1}}\right) \\
&= \frac{\left(2x^1, \ldots, 2x^n, (|\sigma(x)|^2 - 1)(1 - x^{n+1})\right)}{(|\sigma(x)|^2 + 1)(1 - x^{n+1})} \\
&= \frac{\left(2x^1, \ldots, 2x^n, (1 - x^{n+1}) - (1 - x^{n+1})\right)}{\left(\frac{1-(x^{n+1})^2}{(1-x^{n+1})^2} + 1\right)(1 - x^{n+1})} \\
&= \frac{(2x^1, \ldots 2x^n, 2x^{n+1}}{\frac{1-(x^{n+1})^2}{1-x^{n+1}} + 1 - x^{n+1}} \\
&= \frac{(2x^1, \ldots 2x^n, 2x^{n+1}}{\left(\frac{1-(x^{n+1})^2+(1-x^{n+1})^2}{1-x^{n+1}}\right)} \\
&= \frac{(2x^1, \ldots 2x^{n+1})}{\left(\frac{2-2x^{n+1}}{1-x^{n+1}}\right)} \\
&= \frac{(2x^1, \ldots 2x^{n+1})}{2} \\
&= (x^1, \ldots, x^{n+1}) \\
&= x
\end{aligned}
$$

Thus $\sigma^{-1} \circ \sigma$ is the identity on $S^n \setminus \{N\}$. Now we will show that $\sigma \circ \sigma^{-1}$ is the identity in its domain. Let $x = (x^1, \dots x^n) \subset \mathbb{R}^n \setminus \{0\}$. Then

$$\begin{aligned}
\sigma \circ \sigma^{-1}(x) &= \sigma\left(\frac{(2x^1, \dots, 2x^n, |x|^2 - 1)}{|x|^2 + 1}\right) \\
&= \frac{(2x^1, \dots, 2x^n)}{(|x|^2 + 1)(1 - \frac{|x|^2 - 1}{|x|^2 + 1})} \\
&= \frac{(2x^1, \dots, 2x^n)}{|x|^2 + 1 - (|x|^2 - 1)} \\
&= \frac{(2x^1, \dots, 2x^n)}{2)} \\
&= (x^1, \dots x^n) \\
&= x
\end{aligned}$$

Thus $\sigma \circ \sigma^{-1}$ is the identity on $\mathbb{R}^n \setminus \{0\}$. Hence σ is a bijection. □

Proposition 0.6 (Exercise 1.17c)**.** *The atlas consisting of the two charts $\sigma, \tilde{\sigma}$ defines a smooth structure on S^n.*

Proof. To show this, we just need to compute the transition map $\tilde{\sigma} \circ \sigma^{-1} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$.

$$\begin{aligned}
\tilde{\sigma} \circ \sigma^{-1}(u^1, \dots u^n) &= \tilde{\sigma}\left(\frac{(2u^1, \dots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}\right) \\
&= -\sigma\left((-1)\left(\frac{(2u^1, \dots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}\right)\right) \\
&= -\sigma\left(\frac{(2u^1, \dots, 2u^n, |u|^2 - 1)}{-|u|^2 - 1}\right) \\
&= -\frac{(2u^1, \dots, 2u^n)}{(|u|^2 + 1) + (|u|^2 - 1)} \\
&= \frac{(2u^1, \dots, 2u^n)}{2|u|^2} \\
&= \frac{u}{|u|^2}
\end{aligned}$$

Thus this transition map is a diffeomorphism, with itself being the inverse, because

$$(\tilde{\sigma} \circ \sigma^{-1}) \circ (\tilde{\sigma} \circ \sigma^{-1})(u) = \tilde{\sigma} \circ \sigma^{-1}\left(\frac{u}{|u|^2}\right) = \frac{\frac{u}{|u|^2}}{\left|\frac{u}{|u|^2}\right|^2} = \frac{\frac{u}{|u|^2}}{\frac{1}{|u|^2}} = u$$

Thus $\sigma, \tilde{\sigma}$ are compatible charts that cover S^n, so they are a smooth atlas. By Proposition 1.17, we can extend this atlas to a maximal smooth atlas, which give a smooth structure on S^n. □

Proposition 0.7 (Exercise 1-7d)**.** *The smooth structure on S^n induced by the stereographic projection (and the projection excluding the south pole) is the same as the structure induced by the charts $\{U_i^{\pm}\}$ given in Example 1.31.*

Proof. We just need to show that the union of these two smooth atlases is a smooth atlas; that is, we need to show that the stereographic projection σ and the other projection $\tilde{\sigma}$ are compatible with the charts $\{U_i^{\pm}, \phi_i^{\pm}\}$. To do this, we need to show that the transition maps $\sigma \circ (\phi_i^{\pm})^{-1}, \phi_i^{\pm} \circ \sigma^{-1}, \tilde{\sigma} \circ (\phi_i^{\pm})^{-1}$, and $\phi_i^{\pm} \circ \tilde{\sigma}^{-1}$ are all smooth. We will just show that these are smooth for the charts U_i^+, but essentially the same calculations hold for U_i^-.

First, let $x = (x^1, \dots x^n) \in \phi_i^{\pm}(U_i^+ \cap S^n \setminus N)$.

$$\begin{aligned}\sigma \circ (\phi_i^{\pm})^{-1}(x^1, \dots, x^n) &= \sigma(x^1, \dots, x^{i-1}, (1 - |x|^2)^{1/2}, x^i, \dots x^n) \\ &= \frac{(x^1, \dots x^{i-1}, (1 - |x|^2)^{1/2}, x^i, \dots x^{n-1})}{1 - x^n}\end{aligned}$$

This is smooth as long as $x^n \neq 1$, but $x^n \neq 1$ on the domain because the north pole N is excluded. Thus $\sigma \circ (\phi_i^{\pm})^{-1}$ is smooth. Now let $x = \phi_i^{\pm}(U_i^+ \cap S^n \setminus \{S\})$.

$$\begin{aligned}\tilde{\sigma} \circ (\phi_i^{\pm})^{-1}(x^1, \dots x^n) &= -\sigma(-x^1, \dots - x^{i-1}, -(1 - |x|^2)^{1/2}, -x^i, \dots - x^n) \\ &= (-1)\frac{(-x^1, \dots, -x^{i-1}, -(1 - |x|^2)^{1/2}, -x^i, \dots, -x^{n-1})}{1 + x^n}\end{aligned}$$

This is smooth as long as $x^n \neq -1$, but this possibility is excluded because the south pole is not in the domain. Thus $\tilde{\sigma} \circ (\phi_i^{\pm})^{-1}$ is smooth. Now let $x \in \sigma(U_i^+ \cap S^n \setminus \{N\})$.

$$\begin{aligned}\phi_i^{\pm} \circ \sigma^{-1}(x) &= \phi_i^{\pm}\left(\frac{(2x^1, \dots, 2x^n, |x|^2 - 1)}{|x|^2 + 1}\right) \\ &= \frac{(2x^1, \dots, 2x^{i-1}, 2x^{i+1}, \dots, 2x^n, |x|^2 - 1)}{|x|^2 + 1}\end{aligned}$$

This is smooth as long as $|x|^2 \neq -1$, but $|x|^2 \geq 0$. Thus $\phi_i^{\pm} \circ \sigma^{-1}$ is smooth. Finally, let $x \in \tilde{\sigma}(U_i^+ \cap S^n \setminus \{S\})$.

$$\begin{aligned}\phi_i^{\pm} \circ \tilde{\sigma}^{-1}(x) &= \phi_i^{\pm}(-\sigma^{-1}(-x)) \\ &= \phi_i^{\pm}\left((-1)\frac{-2x^1, \dots, -2x^n, |x|^2 - 1}{|x|^2 + 1}\right) \\ &= \frac{(2x^1, \dots, 2x^{i-1}, 2x^{i+1}, \dots, 2x^n, -|x|^2 + 1)}{|x|^2 + 1}\end{aligned}$$

This is also smooth as long as $|x|^2 \neq -1$, it is smooth on its whole domain.

We have shown that each chart $(U_i^+, \phi_i^{\pm})$ is compatible with the charts $(\sigma, S^n \setminus \{N\}), (\tilde{\sigma}, S^n \setminus \{S\})$. These arguments easily extend to show compatibility of $(U_i^-, \phi_i^{\pm})$ with $\sigma, \tilde{\sigma}$. Thus the smooth atlases are compatible, so they induce the same smooth structure by Proposition 1.17b. ☐

Proposition 0.8 (Exercise 1-8)**.** *Let $U \subset S^1$. There exists an angle function $\theta : U \to \mathbb{R}$ satisfying $e^{i\theta(z)} = z$ for $z \in U$ if and only if $U \neq S^1$. Furthermore, when such an angle function exists, (U, θ) is a smooth coordinate chart for S^1 with its standard smooth structure.*

Proof. First suppose that $U = S^1$. Then U is connected and locally path-connected. Let $\pi : \mathbb{R} \to S^1$ be the covering map $t \mapsto e^{2\pi i t}$, and let $\iota : U \hookrightarrow S^1$ be the inclusion map (note that ι is continuous). Then the induced homomorphism $\pi_* : \pi_1(\mathbb{R}) \to \pi_1(S^1)$ is trivial, since it maps the trivial group into $\mathbb{Z}$. Since ι is actually the identity map, it induces an isomorphism $\iota_* : \pi_1(U) \to \pi_1(S^1)$, so $\iota_*(\pi_1(U)) = \mathbb{Z}$.

Hence the inclusion $\iota_*(\pi_1(U)) \subset \pi_*(\pi_1(\mathbb{R}))$ fails, so by Proposition A.78 (Lifting Criterion), there does not exist a continuous function $\theta : U \to \mathbb{R}$ such that $\theta(1) = 1$, and hence no such θ such that $e^{i\theta(1)} = 1$. (If $e^{i\theta(1)} = 1$, then we must have $i\theta(1) = 2\pi k$ for some $k \in \mathbb{Z}$, and $2\pi k$ is can only be a real scalar multiple of i if $k = 0$, hence $\theta(1)$ must be zero to satisfy $e^{i\theta(1)} = 1$.) Thus if $U = S^1$, then no angle function exists.

Now suppose that $U \neq S^1$ is an open subset not equal to S^1. We must construct a continuous function $\theta : U \to \mathbb{R}$. Let $p_0 \in S^1 \setminus U$. Then there exists (not unique) $t_0 \in \mathbb{R}$ such that $e^{it_0} = p_0$. Then for every $p \in S^1 \setminus \{p_0\}$, there exists a unique $t \in (t_0, t_0 + 2\pi)$ such that $e^{it} = p$. Set $\tilde{\theta}(p) = t$, so we have defined a function $\tilde{\theta} : S^1 \setminus \{p_0\} \to \mathbb{R}$, and by construction, $e^{i\tilde{\theta}(p)} = e^{it} = p$. We can then set $\theta = \tilde{\theta}_U : U \to \mathbb{R}$.

We need to show that θ is continuous. Let $(x_n)_{n=1}^\infty$ be a sequence in U with limit $x \in U$, that is, $x_n \to x$. Set $t_n = \theta(x_n)$ and $t = \theta(x)$. Then $x_n = e^{it_n}$ and $x = e^{it}$. Suppose (as an RAA hypothesis) that t_n does not converge to t. Since $t_n \in (t_0, t_0 + 2\pi)$, t_n is a bounded sequence, so by the Bolzano-Weierstrass Theorem, t_n has a convergent subsequence t_{n_k}, with limit $s \neq t$. Since $s \in [t_0, t_0 + 2\pi]$ and $t \in (t_0, t_0 + 2\pi)$ and $s \neq t$, it follows that $e^{is} \neq e^{it}$. But since $t_{n_k} \to s$, we have $e^{it_{n_k}} \to e^{is}$. Then since $x_{n_k} = e^{it_{n_k}}$, we have $x_{n_k} \to e^{is} \neq x$. This is a contradiction, since $x_n \to x$ and x_n has a unique limit (by Exercise A.11). Thus θ is continuous.

Now we show that any continuous angle function $\theta : U \to \mathbb{R}$ is a smooth coordinate chart for S^1 with it standard smooth structure. Let $\theta : U \to \mathbb{R}$ be an angle function, that is, $e^{i\theta(p)} = p$ for $p \in U$. Then θ must be injective, because

$$\theta(p) = \theta(q) \implies e^{i\theta(p)} = e^{i\theta(q)} \implies p = q$$

Furthermore, for $x \in \theta(U)$, $\theta(e^{ix}) = \theta(\cos x + i \sin x) = x$, so $\theta^{-1}(x) = e^{ix}$. Let $\sigma : S^1 \to \mathbb{R}$ be the stereographic projection given by $x_1 + ix_2 = (x_1, x_2) \mapsto \frac{x_1}{1 - x_2}$. Then we compute the transition maps $\sigma \circ \theta^{-1} : \theta(U) \to \sigma(U)$, $\theta \circ \sigma^{-1} : \sigma(U) \to \theta(U)$.

$$\sigma \circ \theta^{-1}(x) = \sigma(\cos x + i \sin x) = \frac{\cos x}{1 - \sin x}$$
$$\theta \circ \sigma^{-1}(x) = \theta\left(\frac{(2x, x^2 - 1)}{(x^2 + 1}\right) = \theta\left(\frac{2x}{x^2 + 1} + i\frac{x^2 - 1}{x^2 + 1}\right) = \tan^{-1}\left(\frac{x^2 - 1}{2x}\right)$$

Both of these are diffeomorphisms on $\theta(U) \subset (t_0, t_0 + 2\pi)$, hence θ is a smooth coordinate chart for S^1 with its standard smooth structure. □

Lemma 0.9 (for Exercise 1-9)**.** *The natural projection* $\pi : \mathbb{C}^{n+1} \to \mathbb{CP}^n$ *is an open map.*

Proof. Let $U \subset \mathbb{C}^{n+1}$ be open. First we claim that for $\lambda \in \mathbb{C}$ with $\lambda \neq 0$, the dilation λU, defined as

$$\lambda U = \{\lambda u : u \in U\}$$

is open. Let $z \in \lambda U$. Then $z = \lambda\omega$ for some $\omega \in U$. Since U is open, there exists $\epsilon > 0$ such that $B(\omega, \epsilon) \subset U$. We claim that $B(z, |\lambda|\epsilon) \subset \lambda U$. To see this, let $c \in B(z, |\lambda|\epsilon)$, so then

$$|c - z| = |c - \lambda\omega| = |\lambda(c/\lambda - \omega)| = |\lambda||c/\lambda - \omega| < |\lambda|\epsilon \implies |c/\lambda - \omega| < \epsilon$$

Thus

$$c/\lambda \in B(\omega, \epsilon) \subset U \implies \lambda(c/\lambda) = c \subset \lambda U$$

so we establish $B(z, |\lambda|\epsilon) \subset \lambda U$, and thus λU is open. Now we claim that

$$\pi^{-1}(\pi(U)) = \bigcup_{\lambda \in \mathbb{C}\setminus\{0\}} \lambda U$$

Let $z \in \pi^{-1}(\pi(U))$. Then $\pi(z) = \pi(\omega)$ for some $\omega \in U$, and thus $z = \lambda\omega$ for some λ, hence $\pi^{-1}(\pi(U))$ is contained in the union of all λU. Now suppose that $z \in \lambda U$. Then $z = \lambda\omega$ for some $\omega \in U$, so $\pi(z) = \pi(\omega)$, so $z \in \pi^{-1}(\pi(\omega))$, hence $z \in \pi^{-1}(\pi(U))$. Thus we have two way containment, so these sets are equal.

We already showed that each λU is open, so the union is open. Hence $\pi^{-1}(\pi(U))$ is open for every open $U \subset \mathbb{C}^{n+1}$. Since π is continuous, $\pi^{-1}(X)$ is open if and only if X is open, so $\pi^{-1}(\pi(U))$ open implies $\pi(U)$ open. Hence $\pi(U)$ is open for every $U \subset \mathbb{C}^{n+1}$ open, so π is an open map. □

Proposition 0.10 (Exercise 1-9)**.** $\mathbb{CP}^n$ *is a compact* $2n$*-dimensional topological manifold, and we can give it a smooth structure.*

Proof. Let $\pi : \mathbb{C}^{n+1} \to \mathbb{CP}^n$ be the natural projection. First $\mathbb{CP}^n$ is compact because it is the image of S^{2n+1} under π. Since π is continuous, and S^{2n+1} is compact, its image is compact under π.

Showing that $\mathbb{CP}^n$ is Hausdorff is beyond the machinery we have so far developed in class. I invoke a theorem of Bourbaki: If G is a compact Hausdorff group and X is a locally compact Hausdorff space, such that G acts continuously on X, then the orbit space X/G is Hausdorff. I assert that $(\mathbb{C} \setminus \{0\}, *)$ is a compact Hausdorff group, and $\mathbb{C}^{n+1}$ is a locally compact Hausdorff space, and $\mathbb{CP}^n$ is the orbit space $\mathbb{C}^{n+1}/(\mathbb{C} \setminus \{0\}, *)$. Hence $\mathbb{CP}^n$ is Hausdorff.

Now we show that $\mathbb{CP}^n$ is second-countable. We know that $\mathbb{C}^{n+1}$ is second-countable, so it has a countable basis $\mathcal{B}$. As shown in the previous lemma, the projection $\pi : \mathbb{C}^{n+1} \to \mathbb{CP}^n$ is an open map. It is also continuous and surjective, so by Lemma 0.2, $\pi(\mathcal{B})$ is a countable basis for $\mathbb{CP}^n$.

Now we show that $\mathbb{CP}^n$ is locally Euclidean of dimension $2n$. For $i = 1, \dots n+1$, let $\tilde{U}_i \subset \mathbb{C}^{n+1}$ be the set

$$\tilde{U}_i = \{(z^1, \dots z^{n+1} : z^i \neq 0\}$$

and define $U_i = \pi(\tilde{U}_i) \subset \mathbb{CP}^n$. Because $\tilde{U}_i$ is a saturated open set, U_i is open and $\pi|_{\tilde{U}_i} : \tilde{U}_i \to U_i$ is a quotient map. Define $\phi_i : U_i \to \mathbb{C}^n$ by

$$\phi_i[z] = \phi_i[z^1, \dots z^{n+1}] = \left(\frac{z^1}{z^i}, \dots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \dots \frac{z^{n+1}}{z^i}\right)$$

The map ϕ_i is well defined, because $\phi_i[az] = \phi_i[z]$ for $a \in \mathbb{C}\setminus\{0\}$, as the following calculation shows.

$$\phi_i[az] = \phi_i[az^1, \dots az^{n+1}] = \left(\frac{az^1}{az^i}, \dots, \frac{az^{i-1}}{az^i}, \frac{az^{i+1}}{az^i}, \dots \frac{az^{n+1}}{az^i}\right) = \phi_i[z]$$

Furthermore, $\phi_i \circ (\pi|_{\tilde{U}_i}) : \tilde{U}_i \to \mathbb{C}^n$ is continuous, so ϕ_i is continuous by Theorem A.27. Actually, ϕ_i is a homeomorphism, because it has the continuous inverse

$$\phi_i^{-1}(z^1, \dots z^n) = [z^1, \dots z^{i-1}, 1, z^i, \dots z^n]$$

To verify that these are inverses, notice that

$$\begin{aligned}
\phi_i \circ \phi_i^{-1}(z^1, \dots z^n) &= \phi[z^1, \dots z^{i-1}, 1, z^i, \dots z^n] = (z^1, \dots z^{i-1}, z^i, \dots z^n) \\
\phi_i^{-1} \circ \phi_i[z^1, \dots z^{n+1}] &= \phi_i^{-1}\left(\frac{z^1}{z^i}, \dots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \dots \frac{z^{n+1}}{z^i}\right) \\
&= \left[\frac{z^1}{z^i}, \dots, \frac{z^{i-1}}{z^i}, \frac{z^i}{z^i}, \frac{z^{i+1}}{z^i}, \dots \frac{z^{n+1}}{z^i}\right] \\
&= \left[\frac{(z^1, \dots z^{n+1})}{z^i}\right] \\
&= [z^1, \dots z^{n+1}]
\end{aligned}$$

Since the sets $U_1, \dots U_{n+1}$ cover $\mathbb{CP}^n$, this shows that every point in $\mathbb{CP}^n$ has a neighborhood U_i that is homeomorphic to $\phi(U_i) \subset \mathbb{C}^n$. But the identification

$$\psi : (x^1 + iy^1, \dots, x^n + iy^n) \to (x^1, y^1, \dots x^n, y^n)$$

is a homeomorphism between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$. Let $W_i = \phi_i \circ \pi(U_i)$. Then $\phi_i \circ \psi : U_i \to W_i \subset \mathbb{R}^{2n}$ is a homeomorphism. Since the collection of U_i cover $\mathbb{CP}^n$, this shows that $\mathbb{CP}^n$ is locally Euclidean of dimension $2n$.

Now we show how to put a smooth structure on $\mathbb{CP}^n$. As shown above, $(U_i, \phi_i \circ \psi)$ are charts for $\mathbb{CP}^n$. We just need to show that the transition map $(\phi_i \circ \psi) \circ (\phi_j \circ \psi)^{-1}$ is a diffeomorphism.

$$(\phi_i \circ \psi) \circ (\phi_j \circ \psi)^{-1}) = \phi \circ \psi \circ \psi^{-1} \circ \phi_j^{-1} = \phi_i \circ \phi_j^{-1}$$

This composition, $\phi_i \circ \phi_j^{-1}$, is shown to be a diffeomorphism in Example 1.33 of Lee. $\square$